A Convex Optimization Approach to Radiation Treatment Planning with Dose Constraints

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Outline

Radiation Therapy

- Patient is scanned to obtain 3D image of anatomy, delineated into structures consisting of discrete voxels
- Clinician identifies planning target volumes (PTVs) and organs-at-risk (OARs)
- Clinician prescribes radiation dose for each PTV
- Patient is placed on couch, and radiation beams are delivered via a linear accelerator

Medical Linear Accelerator

Figure: Varian Linear Accelerator at Central Vermont Medical Center

Treatment Planning

- Radiation traveling through body damages both diseased and healthy tissue
- Trade-off between irradiating PTVs and sparing OARs

Treatment Planning

- Radiation traveling through body damages both diseased and healthy tissue
- Trade-off between irradiating PTVs and sparing OARs

Given beam positions and shapes, what beam intensities

- Deliver "close" to the prescribed dose for each PTV, and
- Minimize radiation to OARs and healthy tissue?

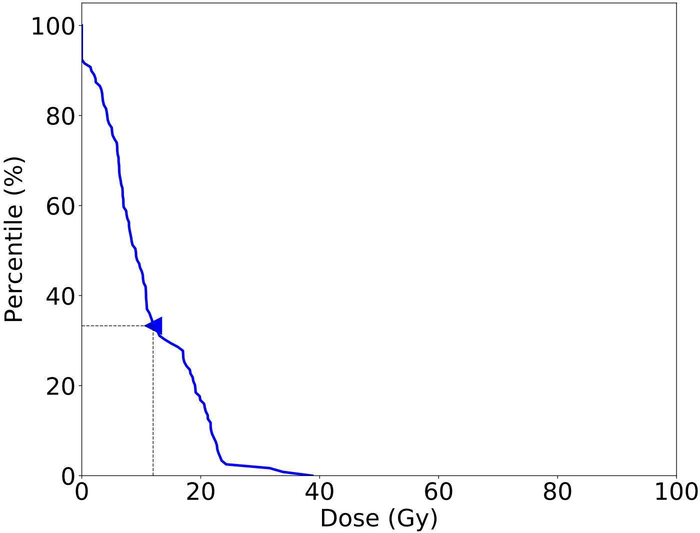
Dose-Volume Histogram (DVH) Constraints

- Different anatomical structures react differently to radiation
- Need to control **dose distribution** over a structure
- We do this by imposing a DVH constraint:

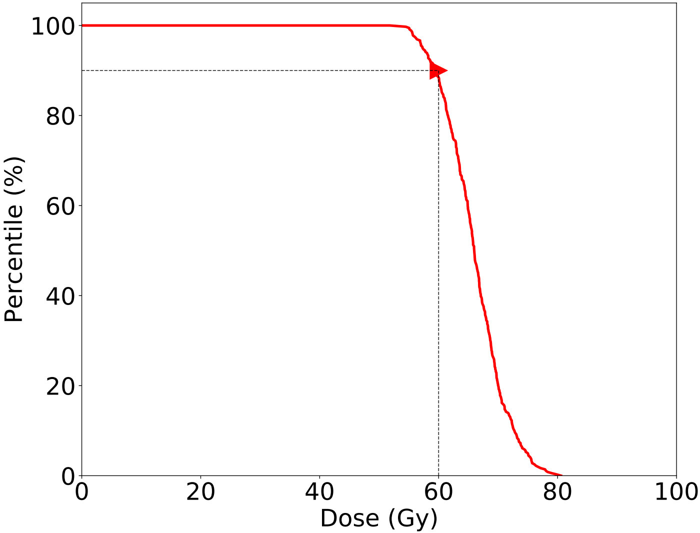
  At most (least) p% of structure receives over b Gy

- Ex: $D(33) \leq 12$ on an OAR means at most 33% of the organ receives over 12 Gy.

Upper DVH Constraint: $D(33) \leq 12$



Lower DVH Constraint: $D(90) \geq 60$



Previous Work

- Linear and quadratic programming (Shepard et al, 1999)
- Multi-objective optimization (Hamacher & Küfer, 2002)
- Volumetric dose penalty with local search (Ehrgott et al, 2008)
- Mixed-integer programming (Lee, Fox & Crocker, 2000)
- CVaR approximation (Rockafellar & Uryasev, 2000)
- Constraints on dose moments (Zarepisheh et al, 2013)

Outline

Convex Model

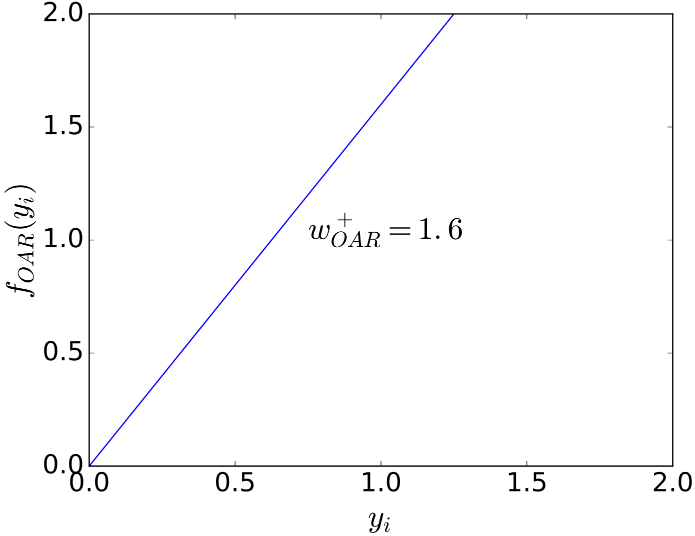
$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & f(y) \\ \text{subject to} & Ax = y, \quad x \geq 0 \end{array}$$

- $x \in \mathbf{R}^n_+$ beam intensities, $y \in \mathbf{R}^m_+$ voxel doses
- A is (m voxels) $\times$ (n beams) dose influence matrix
- $f : \mathbf{R}^n \to \mathbf{R}$ is sum of piecewise linear functions

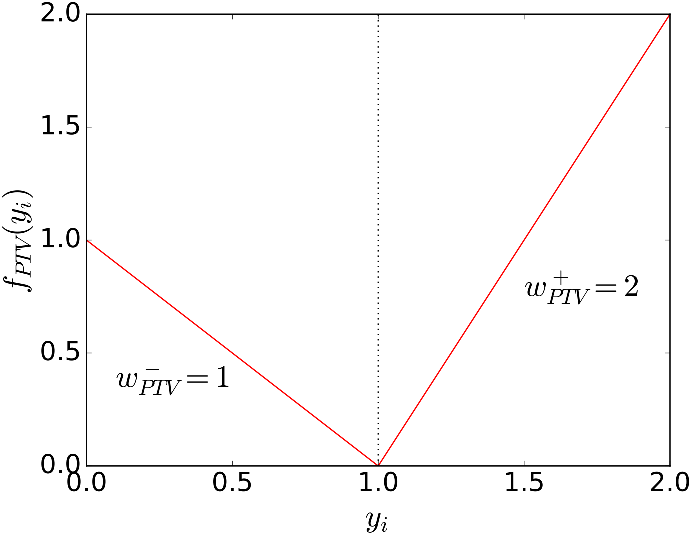
$$f_s(y_i) = w_s^- \underbrace{(y_i - d_s)_-}_{\text{underdose}} + w_s^+ \underbrace{(y_i - d_s)_+}_{\text{overdose}},$$

where d_s is prescribed dose and (w_s^-, w_s^+) are penalty weights for structure s

Penalty Function (OAR)



Penalty Function (PTV)



DVH Constraint Model

- Recall the (upper) DVH constraint $D_s(p, y) \leq b$:

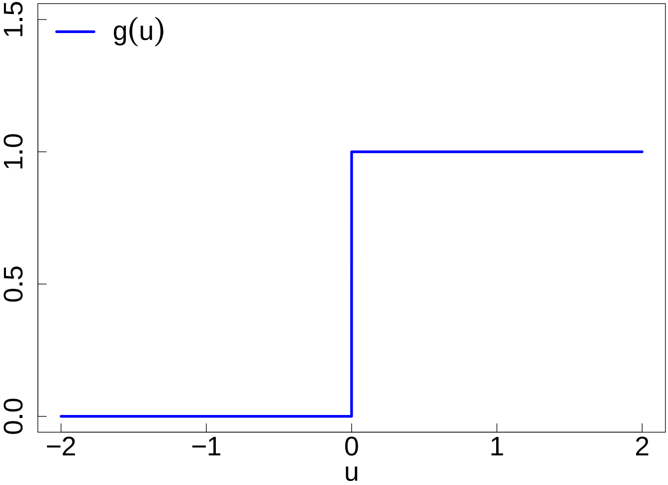
  At most p% of structure s receives over b Gy

- Let $\mathcal{V}_s$ be the set of voxels for structure s
- If $\phi_s(p) = (p\%$ of voxels in $s)$, we can write this as

$$v_s(y, b) := \sum_{i \in \mathcal{V}_s} \mathbf{1}\{y_i \geq b\} = \sum_{i \in \mathcal{V}_s} g(y_i - b) \leq \phi_s(p),$$

where $g(u) = \mathbf{1}\{u \geq 0\}$

Indicator Function

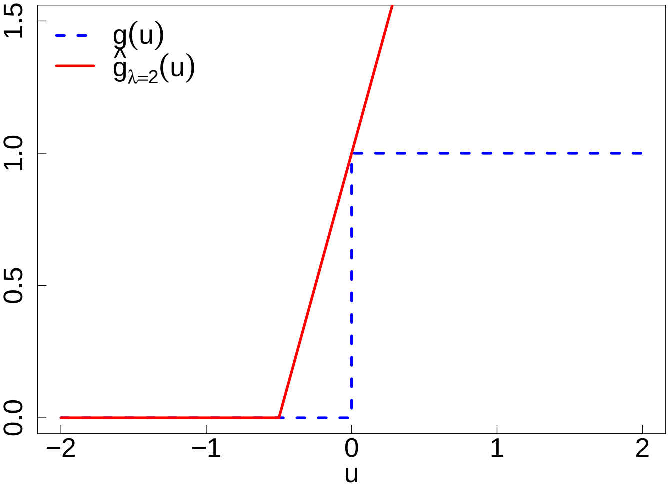


Convex Restriction

- DVH constraints are **not** convex!
- Replace $g(\cdot)$ with convex hinge loss

$$\hat{g}_\lambda(u) = \max(1 + \lambda u, 0) \quad \text{for } \lambda > 0$$

Hinge Loss vs. Indicator



Convex Restriction

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$$\hat{g}_\lambda(u) = \max(1 + \lambda u, 0) \quad \text{for } \lambda > 0$$

- Since $g(u) \leq \hat{g}_\lambda(u)$, this provides restriction

$$\sum_{i \in \mathcal{V}_s} g(y_i - b) \leq \sum_{i \in \mathcal{V}_s} \hat{g}_\lambda(y_i - u) \leq \phi_s(p)$$

Restricted Problem

- By defining $\alpha := \frac{1}{\lambda} > 0$, restriction can be written as

$$\hat{D}_s(p, y, b, \alpha) = \sum_{i \in \mathcal{V}_s} (\alpha + (y_i - b))_+ - \alpha \phi_s(p) \leq 0$$

- Convex restricted problem is

$$\begin{array}{ll} \underset{x,y,\alpha}{\text{minimize}} & f(y) \\ \text{subject to} & Ax = y, \quad x \geq 0, \quad \alpha \geq 0 \\ & \hat{D}_s(p, y, b, \alpha) \leq 0 \end{array}$$

with variables $x \in \mathbf{R}^n_+$, $y \in \mathbf{R}^m_+$, and $\alpha \in \mathbf{R}_+$

Outline

Two-Pass Refinement

- $D_s(p, y) \leq b$ iff $y_i \leq b$ for at least $(100 - p)\%$ of voxels in s
- Refinement selects voxels to precisely bound using heuristic
- Solution with precise bound always satisfies DVH constraint

Two-Pass Refinement

- **First pass:** Solve convex restricted problem for (x^*, y^*, α^*)
- Compute underdose margins $\xi_i^* = b - y_i^*$
- Identify $\lceil \phi_s(100 - p) \rceil$ largest margins and include their indices in $\mathcal{Q}$
- **Second pass:** Solve for (x^{**}, y^{**}) in

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & f(y) \\ \text{subject to} & y = Ax, \quad x \geq 0 \\ & y_i \leq b, \quad \forall i \in \mathcal{Q} \end{array}$$

using (x^*, y^*) as a warm start

DVH Constraints with Slack

- Convex restricted problem may be infeasible even if original problem is feasible
- Add slack variable $\delta \in \mathbf{R}_+$ so restriction always feasible

$$
\begin{array}{ll}
\underset{x,y,\alpha,\delta}{\text{minimize}} & f(y) \\
\text{subject to} & y = Ax, \quad x \geq 0, \quad \alpha \geq 0, \quad \delta \geq 0 \\
& \hat{D}_s(p, y, b + \delta, \alpha) \leq 0
\end{array}
$$

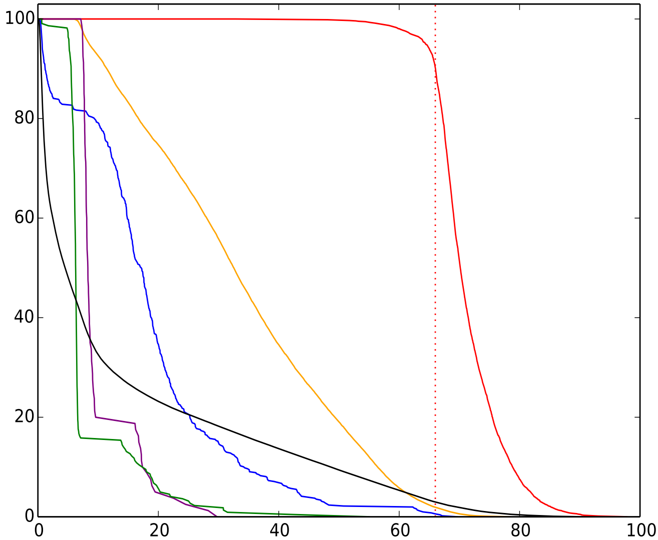
- For second pass, use slack margins $\xi_i^* = b + \delta^* - y_i$

Outline

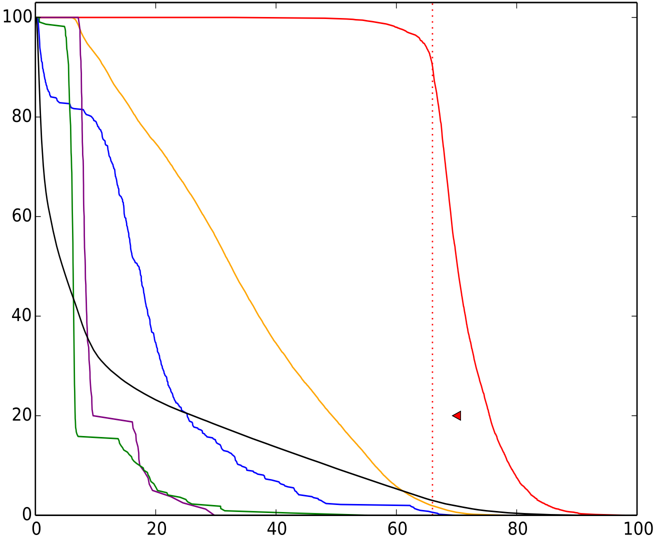
Head and Neck Case

- 4-arc VMAT aperture re-weighting case
- 270,000 voxels $\times$ 360 beams
- 17 structures: PTV (66 Gy), OARs, generic body voxels

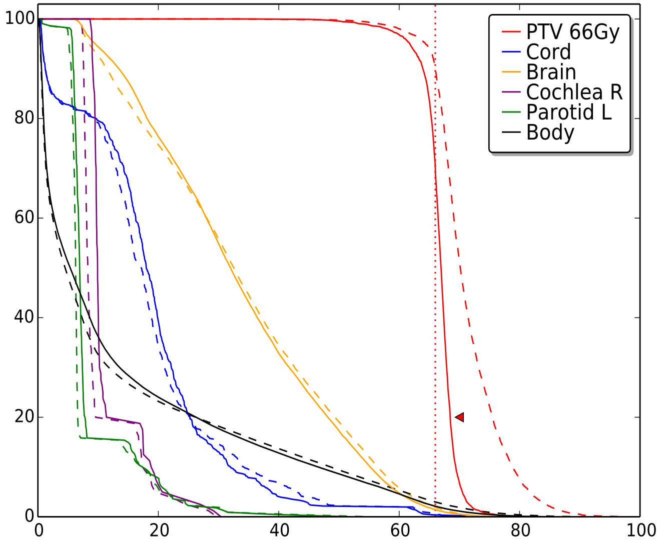
Unconstrained Plan



Add $D(20) \leq 70$ Gy on PTV

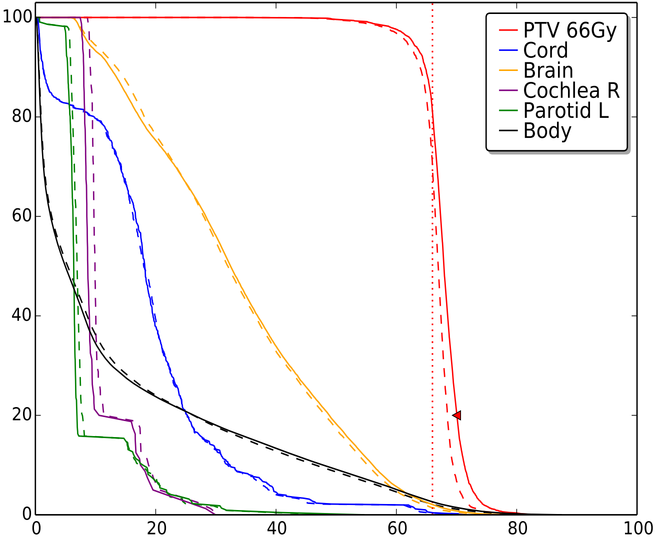


First Pass



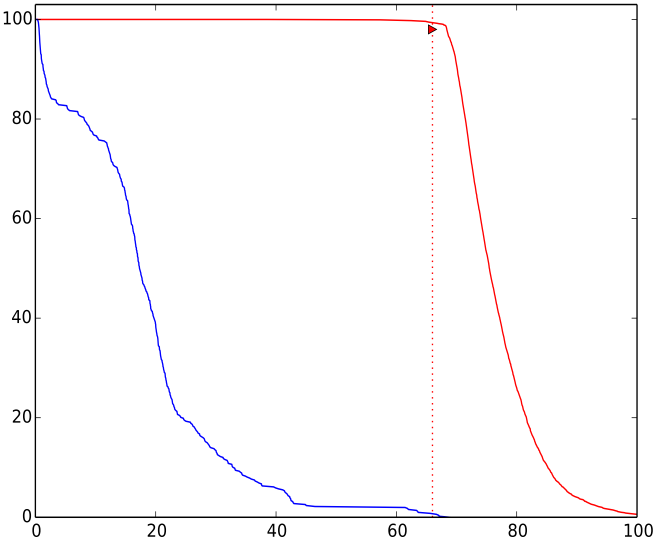
PTV 66Gy
Cord
Brain
Cochlea R
Parotid L
Body
100
80
60
40
20
0
0
20
40
60
80
100

Second Pass

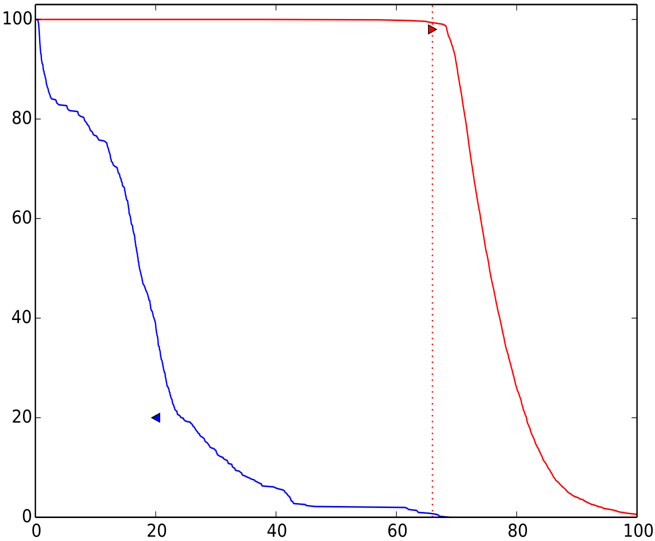


PTV 66Gy
Cord
Brain
Cochlea R
Parotid L
Body
0
20
40
60
80
100
0
20
40
60
80
100

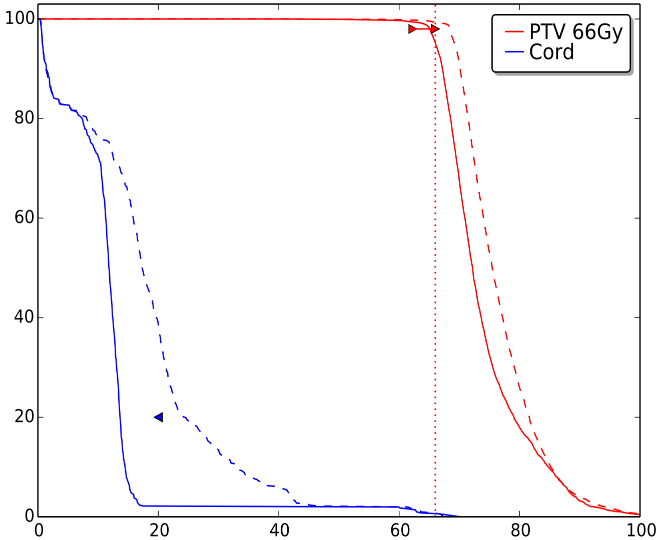
$D(98) \geq 66$ **Gy on PTV, No Slack**



Add $D(20) \leq 20$ Gy on OAR



Plan with Slackened PTV Constraint



Conclusion

- First principled method for handling DVH constraints
- Python library `ConRad` with intuitive interface
- `http://stanford.edu/~boyd/papers/conrad.html`